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Note

On spanning connected graphs

Cheng-Kuan Lin[a,*], Hua-Min Huang[b], Jimmy J.M. Tan[a], Lih-Hsing Hsu[c]

[a] *Department of Computer Science, National Chiao Tung University, Hsinchu, Taiwan 30010, ROC*
[b] *Department of Mathematics, National Central University, Chungli, Taiwan 32001, ROC*
[c] *Department of Computer Science and Information Engineering, Providence University, Taichung, Taiwan 43301, ROC*

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Abstract

A *k-container* $C(u, v)$ of G between u and v is a set of k internally disjoint paths between u and v. A k-container $C(u, v)$ of G is a k^**-container* if the set of the vertices of all the paths in $C(u, v)$ contains all the vertices of G. A graph G is k^**-connected* if there exists a k^*-container between any two distinct vertices. Therefore, a graph is 1^*-connected (respectively, 2^*-connected) if and only if it is hamiltonian connected (respectively, hamiltonian). In this paper, a classical theorem of Ore, providing sufficient conditional for a graph to be hamiltonian (respectively, hamiltonian connected), is generalized to k^*-connected graphs.

Keywords: Hamiltonian connected; Hamiltonian; Ore Theorem; Menger Theorem

1. Introduction and definitions

For the graph definition and notation we follow [3]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. We use $n(G)$ to denote $|V|$. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The *induced subgraph* $G[H]$ is a subgraph of G where $V(G[H]) = V(H)$ and $E(G[H]) = \{(u, v) \mid (u, v) \in E(G)$ and $u, v \in V(H)\}$. Two vertices u and v are *adjacent* if (u, v) is an edge of G. Let v be a vertex of G and H be a subgraph of G. The *neighborhood* of u respective to H, denoted by $N_H(u)$, is $\{v \in V(H) \mid (u, v) \in E(G)\}$. The *degree* $d_H(u)$ of a vertex u respective to H is the number of edges between u and $V(H)$. The *minimum degree* of G, written $\delta(G)$, is $\min\{d_G(x) \mid x \in V\}$. A *path* is a sequence of vertices represented by $\langle v_0, v_1, \ldots, v_k\rangle$ with no repeated vertex, and (v_i, v_{i+1}) is an edge of G for all $0 \leqslant i \leqslant k - 1$. We also write the path $\langle v_0, v_1, \ldots, v_k\rangle$ as $\langle v_0, \ldots, v_i, Q, v_j, \ldots, v_k\rangle$, where Q is a path form v_i to v_j. A path is a *hamiltonian path* if it contains all the vertices of G. A graph G is *hamiltonian connected* if, for any two distinct vertices of G, there exists a hamiltonian path joining those two vertices. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G. A graph is *hamiltonian* if it has a hamiltonian cycle. We use $G \cup H$ to denote the disjoint union of graph G and graph H. Moreover, we use $G \vee H$ to denote the graph obtained from $G \cup H$ by joining all the edges with one vertex in G and the other vertex in H. Let u and v be two nonadjacent vertices of G, we use $G + uv$ to denote the graph obtained from G by adding the edge (u, v).

* Corresponding author.
E-mail address: cklin@cs.nctu.edu.tw (C.-K. Lin).

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A *k-container* $C(u, v)$ of G between u and v is a set of k internally disjoint paths between u and v. In other words, $C(u, v)$ consists of paths $P_1, P_2, \ldots, P_k$ such that $E(P_i) \cap E(P_j) = \emptyset$ and $V(P_i) \cap V(P_j) = \{u, v\}$ for $1 \leqslant i \neq j \leqslant k$. The concept of container is proposed by Hsu [5] to evaluate the performance of communication of an interconnection network. The *connectivity* of G, $\kappa(G)$, is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. It follows from Menger's Theorem [7] that there is a k-container between any two distinct vertices of G if and only if G is k-connected.

In this paper, we are interested in a special type of container. A k-container $C(u, v)$ of G is a k^*-*container* if the set of the vertices of all the paths in $C(u, v)$ contains all the vertices of G. A graph G is k^*-*connected* if there exists a k^*-container between any two distinct vertices. A 1^*-connected graph except K_1 and K_2 is 2^*-connected. A 1^*-connected graph is actually a hamiltonian connected graph. Moreover, a 2^*-connected graph is a hamiltonian graph. Thus, the concept of k^*-connected graph is a hybrid concept of connectivity and hamiltonicity. The study of k^*-connected graph is motivated by the globally 3^*-connected graphs proposed by Albert et al. [1]. A globally 3^*-connected graph is a cubic graph that is w^*-connected for all $1 \leqslant w \leqslant 3$. Recently, Lin et al. [6] proved that the pancake graph P_n is w^*-connected for any w with $1 \leqslant w \leqslant n-1$ if and only if $n \neq 3$. Thus, we defined the *spanning connectivity* $\kappa^*(G)$ of a graph G to be the largest integer k such that G is w^*-connected for all $1 \leqslant w \leqslant k$ if G is 1^*-connected graph and undefined otherwise. A graph G is *super spanning connected* if $\kappa^*(G) = \kappa(G)$. The complete graph K_n is super spanning connected, and the pancake graph P_n is super spanning connected if and only if $n \neq 3$.

Let k be a positive integer. In this paper, we have the following results. If there exist two nonadjacent vertices u and v with $d_G(u) + d_G(v) \geqslant n(G) + k$ then G is $(k+2)^*$-connected if and only if $G + uv$ is $(k+2)^*$-connected. Moreover, if there exist two nonadjacent vertices u and v with $d_G(u) + d_G(v) \geqslant n(G) + k$, then G is i^*-connected if and only if $G + uv$ is i^*-connected for $1 \leqslant i \leqslant k+2$. Assume that $d_G(u) + d_G(v) \geqslant n + k$ for all nonadjacent vertices u and v, then G is r^*-connected for every $r \in \{1, 2, \ldots, k+2\}$.

2. Sufficient condition for spanning connected graphs

Ore [8,9], and Bondy and Chvátal [2] proved the following theorem:

Theorem 1 (*Bondy and Chvátal [2], Ore [8,9]*). *Assume that there exist two nonadjacent vertices u and v with $d_G(u)+d_G(v) \geqslant n(G)$ then G is 2^*-connected if and only if $G+uv$ is 2^*-connected. Moreover, $d_G(u)+d_G(v) \geqslant n(G)+1$ then G is 1^*-connected if and only if $G+uv$ is 1^*-connected.*

Lemma 1. *Let k be a positive integer. Suppose that there exist two nonadjacent vertices u and v with $d_G(u) + d_G(v) \geqslant n(G) + k$. Then, for any two distinct vertices x and y, G has a $(k+2)^*$-container between x and y if and only if $G + uv$ has a $(k+2)^*$-container between x and y.*

Proof. If G has a $(k+2)^*$-container between x and y, then clearly $G+uv$ has a $(k+2)^*$-container between x and y. For the other direction, let $C(x, y) = \{P_1, P_2, \ldots, P_{k+2}\}$ be a $(k+2)^*$-container of $G+uv$ between x and y. Suppose that the edge $(u, v) \notin C(x, y)$. Then $C(x, y)$ forms a desired $(k+2)^*$-container of G. Thus, we suppose that $(u, v) \in P_1$. We write P_1 as $\langle x, H_1, u, v, H_2, y\rangle$and write P_i as $\langle x, P_i', y\rangle$ for $2 \leqslant i \leqslant k+2$. (Note that $l(H_1) = 0$ if $x = u$, and $l(H_2) = 0$ if $y = v$.) We set $C_i = \langle x, P_i', y, H_2^{-1}, v, u, H_1^{-1}, x\rangle$ for $2 \leqslant i \leqslant k+2$.

Case 1: $d_{G[C_i]}(u) + d_{G[C_i]}(v) \geqslant n(C_i)$ for some $2 \leqslant i \leqslant k+2$. Without loss of generality, we may assume that $d_{G[C_2]}(u) + d_{G[C_2]}(v) \geqslant n(C_2)$. By Theorem 1, there is a hamiltonian cycle C of the induced subgraph $G[C_2]$. Let $C = \langle x, R_1, y, R_2, x\rangle$. We set $Q_1 = \langle x, R_1, y\rangle$, $Q_2 = \langle x, R_2^{-1}, y\rangle$, and $Q_i = P_i$ for $3 \leqslant i \leqslant k+2$. Then $\{Q_1, Q_2, \ldots, Q_{k+2}\}$ forms a $(k+2)^*$-container of G between x and y.

Case 2: $d_{G[C_i]}(u) + d_{G[C_i]}(v) \leqslant n(C_i) - 1$ for all $2 \leqslant i \leqslant k+2$. Since

$$\begin{aligned}\sum_{i=2}^{k+2}(d_{G[C_i]}(u) + d_{G[C_i]}(v)) &= \sum_{i=2}^{k+2}(d_{G[P_i']}(u) + d_{G[P_1]}(u) + d_{G[P_i']}(v) + d_{G[P_1]}(v))\\ &= \sum_{i=2}^{k+2}(d_{G[P_i']}(u) + d_{G[P_i']}(v)) + (k+1)(d_{G[P_1]}(u) + d_{G[P_1]}(v))\end{aligned}$$

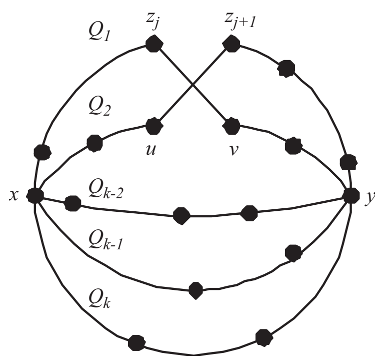

Fig. 1. Illustration for case 2 of Lemma 1.

$$= d_G(u) + d_G(v) + k(d_{G[P_1]}(u) + d_{G[P_1]}(v))$$

$$\geqslant n(G) + k + k(d_{G[P_1]}(u) + d_{G[P_1]}(v))$$

and

$$\begin{aligned}\sum_{i=2}^{k+2}(n(C_i) - 1) &= \sum_{i=2}^{k+2}(n(P_i') + n(P_1)) - (k+1)\\ &= \sum_{i=2}^{k+2} n(P_i') + (k+1)(n(P_1)) - (k+1)\\ &= n(G) + k(n(P_1)) - (k+1),\end{aligned}$$

$n(G) + k + k(d_{G[P_1]}(u) + d_{G[P_1]}(v)) \leqslant n(G) + k(n(P_1)) - (k+1)$. Therefore, $d_{G[P_1]}(u) + d_{G[P_1]}(v) \leqslant n(P_1) - 2$.

We claim that $d_{G[P_i']}(u) + d_{G[P_i']}(v) \geqslant n(P_i') + 2$ for some $2 \leqslant i \leqslant k+2$. Suppose that $d_{G[P_i']}(u) + d_{G[P_i']}(v) \leqslant n(P_i') + 1$ for all $2 \leqslant i \leqslant k+2$. Then

$$\begin{aligned}d_G(u) + d_G(v) &= \sum_{i=2}^{k+2}(d_{G[P_i']}(u) + d_{G[P_i']}(v)) + (d_{G[P_1]}(u) + d_{G[P_1]}(v))\\ &\leqslant \sum_{i=2}^{k+2}(n(P_i') + 1) + n(P_1) - 2\\ &= n(G) + k - 1.\end{aligned}$$

This contradicts the fact that $d_G(u) + d_G(v) \geqslant n + k$.

Without loss of generality, we may assume that $d_{G[P_2']}(u) + d_{G[P_2']}(v) \geqslant n(P_2') + 2$. Obviously, $n(P_2') \geqslant 2$. We write $P_2 = \langle x, z_1, z_2, \ldots, z_r, y\rangle$. Then, there exists $j \in \{1, 2, \ldots, r-1\}$ such that $(z_j, v) \in E(G)$ and $(z_{j+1}, u) \in E(G)$. For otherwise, $d_{G[P_2']}(u) + d_{G[P_2']}(v) \leqslant r + r - (r-1) = r + 1 = n(P_2') + 1$, giving a contradiction. We set $Q_1 = \langle x, z_1, z_2, \ldots, z_j, v, H_2, y\rangle$, $Q_2 = \langle x, H_1, u, z_{j+1}, z_{j+2}, \ldots, z_r, y\rangle$, and $Q_i = P_i$ for $3 \leqslant i \leqslant k+2$. Then $\{Q_1, Q_2, \ldots, Q_{k+2}\}$ forms a k^*-container of G between x and y. See Fig. 1 for an illustration. □

With Lemma 1, we have the following theorem:

Theorem 2. *Assume that k is any positive integer and there exist two nonadjacent vertices u and v with $d_G(u) + d_G(v) \geqslant n(G) + k$. Then G is $(k+2)^*$-connected if and only if $G + uv$ is $(k+2)^*$-connected. Moreover, G is i^*-connected if and only if $G + uv$ is i^*-connected for $1 \leqslant i \leqslant k+2$.*

Theorem 3 (*Ore [9]*). *Assume that $d_G(u) + d_G(v) \geqslant n(G) + 1$ for all nonadjacent vertices u and v of G. Then G is 1^*-connected.*

Theorem 4. *Let k be a positive integer. Assume that $d_G(u) + d_G(v) \geqslant n(G) + k$ for all nonadjacent vertices u and v of G, then G is r^*-connected for every $1 \leqslant r \leqslant k + 2$.*

Proof. By Theorem 3, G is 1^*-connected and 2^*-connected. Let x and y be two distinct vertices in G. Suppose there exists an r^*-container $\{P_1, P_2, \ldots, P_r\}$ of G between x and y for some $2 \leqslant r \leqslant k + 1$. We only need to construct an $(r+1)^*$-container of G between x and y. We have $d_G(y) \geqslant k+2$, for otherwise let $w \notin N_G(y)$then $d_G(y)+d_G(w) \leqslant (k+1)+(n-2) = n+k-1$, which is a contradiction. We can choose a vertex u in $N_G(y) - \{x\}$ such that $(u, y) \notin E(P_i)$ for all $1 \leqslant i \leqslant r$. Without loss of generality, assume that $u \in P_r$ and we write P_r as $\langle x, H_1, u, v, H_2, y\rangle$. We set $Q_i = P_i$ for $1 \leqslant i \leqslant r - 1$, $Q_r = \langle x, H_1, u, y\rangle$, and $Q_{r+1} = \langle x, v, H_2, y\rangle$. Suppose that $(x, v) \in E(G)$. Then $\{Q_1, Q_2, \ldots, Q_{r+1}\}$ forms an $(r + 1)^*$-container of G between x and y. Suppose that $(x, v) \notin E(G)$. Then, $\{Q_1, Q_2, \ldots, Q_{r+1}\}$ forms an $(r + 1)^*$-container of $G + xu$ between x and y. By Lemma 1, there exists an $(r + 1)^*$-container of G between x and y. $\square$

We give an example to show that the above result may not hold for $r = k + 3$. Therefore, our result is optimal. Let K_n be a complete graph with n vertices. We set $G = (K_1 \cup K_b) \vee K_a$ where $a \geqslant 3$ and $b \geqslant 2$. Obviously, $\delta(G) = a$ and $d_G(u) + d_G(v) \geqslant 2a + b - 1$ for any two distinct vertices u and v. Thus, G is not r^*-connected for any $r > a$.

Dirac [4] proved that any graph G with at least three vertices and $\delta(G) \geqslant n(G)/2$ is 2^*-connected. Any graph G with at last four vertices and $\delta(G) \geqslant n(G)/2 + 1$ is 1^*-connected. Obviously, if G is a complete graph then it is super spanning connected. Thus, we consider incomplete graphs.

Theorem 5. *Assume that G is a graph with $n(G)/2+1 \leqslant \delta(G) \leqslant n(G)-2$. Then G is r^*-connected for $1 \leqslant r \leqslant 2\delta(G) - n(G) + 2$.*

Proof. Since $n(G)/2+1 \leqslant \delta(G) \leqslant n(G)-2$, $n(G) \geqslant 6$. Let k be a positive integer and $m \geqslant 3$. Suppose that $n(G) = 2m$ and $\delta(G) = m + k$ for some $m \geqslant 3$ and $1 \leqslant k \leqslant m - 2$. Then $d_G(u) + d_G(v) \geqslant 2\delta(G) = 2m + 2k$. By Theorem 4, G is r^*-connected for $1 \leqslant r \leqslant 2k + 2$. Suppose that $n(G) = 2m + 1$ and $\delta(G) = m + 1 + k$ for some $m \geqslant 3$ and $1 \leqslant k \leqslant m - 2$. We have $d_G(u) + d_G(v) \geqslant 2\delta(G) = 2m + 2 + 2k$. By Theorem 4, G is r^*-connected for $1 \leqslant r \leqslant 2k + 3$. $\square$

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